Scaling theory of temporal correlations and size-dependent fluctuations in the traded value of stocks

Zoltán Eisler* and János Kertész†
Department of Theoretical Physics, Budapest University of Technology and Economics, Budapest, Hungary
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Records of the traded value f_i of stocks display fluctuation scaling, a proportionality between the standard deviation σ_i and the average $\langle f_i \rangle$: $\sigma_i \propto \langle f_i \rangle^{\alpha}$, with a strong time scale dependence $\alpha(\Delta t)$. The nontrivial (i.e., neither 0.5 nor 1) value of α may have different origins and provides information about the microscopic dynamics. We present a set of stylized facts and then show their connection to such behavior. The functional form $\alpha(\Delta t)$ originates from two aspects of the dynamics: Stocks of larger companies both tend to be traded in larger packages and also display stronger correlations of traded value. The results are integrated into a general framework that can be applied to a wide range of complex systems.

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I. INTRODUCTION

Research concerning the forces that govern stock markets is largely helped by the abundant availability of data on trading activity [1,2]. Recently an increasing number of complex systems have been studied in a similar, data-oriented way [3,4]. Examples include records of information flow through routers on the Internet or of webpage visitations [5,6]. The means to extract information from such multichannel data are generally limited to the system-wide distributions of various quantities and cross-correlation measurements. Although these approaches have been very fruitful in various areas, they often fail to provide information on the mechanisms that govern the observed internal processes. On the grounds of a recently discovered set of empirical facts regarding stock market trading activity [7–10], we test a tool that addresses these questions. It is based on an empirical scaling law that appears to hold for most systems. It connects the fluctuations σ_i and the average $\langle f_i \rangle$ of the activity of constituents by a *fluctuation scaling* law:

$$\sigma_i \propto \langle f_i \rangle^{\alpha}.$$

In several studies, the value of α is used as a proxy of the dominant factors of internal dynamics [5–7]. $\alpha = 0.5$ is said to characterize equilibrium systems, while $\alpha = 1$ is considered a consequence of strong external driving. While the general idea works well in a number of settings, we find that α shows a much richer behavior than previously anticipated. When calculated for systems with strong temporal correlations, α becomes time scale dependent. Moreover, even in the lack of correlations, $\alpha > 0.5$ is possible. The aim of this paper is twofold: We present a detailed theory for the emergence of fluctuation scaling, which explains such anomalies, and we also apply this theory to explain phenomena observed in finance.

Section II introduces notations and our set of stock market data. Section III presents some stylized facts regarding stock market trading activity. Section IV describes the concept of fluctuation scaling that connects all those observations, including the previously identified two universality classes $\alpha = 0.5$ and $\alpha = 1$. Then, we deal with a mechanism that explains how stock markets can display a nonuniversal value of $\alpha \approx 0.68$. Finally, we describe how dynamical correlations are reflected in the time scale dependence of the exponent α.

II. NOTATION AND DATA

For our analysis of financial data, it is necessary to give a few definitions. For a time window size Δt, one can write the total traded value of the ith stock at time t in the form

$$f_i^{\Delta t}(t) = \sum_{n, t_i(n) \in [t, t+\Delta t]} V_i(n), \quad (1)$$

where $t_i(n)$ is the time of the nth transaction of stock i. The so-called tick-by-tick data are denoted by $V_i(n)$; this is the value traded in transaction n. It can be calculated as the product of the price p and the traded volume $\tilde{V}$:

$$V_i(n) = p_i(n) \tilde{V}_i(n). \quad (2)$$

The price serves as a weighting factor to make the comparison of different stocks possible, while this definition also eliminates the effect of stock splits.

As the source of empirical data, we used the TAQ database [11], which records all transactions of the New York Stock Exchange (NYSE) and NASDAQ for the years 2000–2002. Our sample was restricted to those 2647 stocks for NYSE and 4039 for NASDAQ that were continuously traded in the period. We divided the data by the well-known U-shaped daily pattern of traded volumes, similarly to Ref. [7].

Finally, note that we use 10-base logarithms throughout the paper to ease the understanding of financial data.

*Electronic address: eisler@maxwell.phy.bme.hu
†Also at Laboratory of Computational Engineering, Helsinki University of Technology, Espoo, Finland.

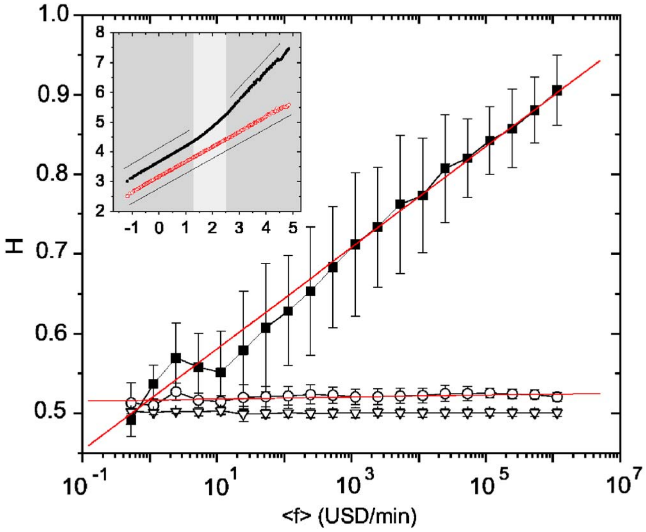

FIG. 1. (Color online) Behavior of the Hurst exponents $H(i)$ for NYSE stocks in the period 2000–2002. For short time windows (○), all signals are nearly uncorrelated, $H(i)\approx 0.51-0.52$. The fitted slope is $\gamma_t(\Delta t<20\text{ min})=0.001\pm 0.002$. For larger time windows (■), the strength of correlations depends logarithmically on the mean trading activity of the stock, $\gamma_t(\Delta t>300\text{ min})=0.06\pm 0.01$. Shuffled data (▽) display no correlations; thus, $H_{\text{shuff}}(i)=0.5$, which also implies $\gamma_t=0$. The inset shows the log σ-log Δt scaling plot for General Electric (GE). The darker shaded intervals have well-defined Hurst exponents; the crossover is indicated with a lighter background. The slopes corresponding to Hurst exponents are 0.53 and 0.93; the slope for shuffled data is 0.51. Shuffled points were shifted vertically for better visibility.

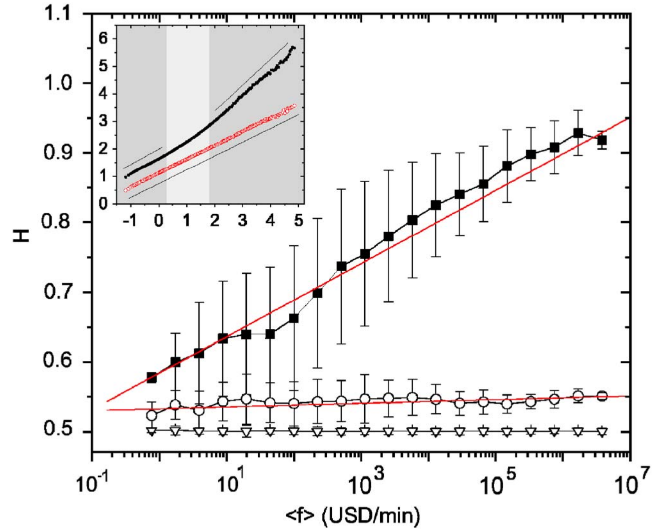

FIG. 2. (Color online) Behavior of the Hurst exponents $H(i)$ for NASDAQ stocks in the period 2000–2002. For short time windows (○), all signals are nearly uncorrelated, $H\approx 0.52-0.53$. The fitted slope is $\gamma_t(\Delta t<2\text{ min})=0.003\pm 0.002$. For larger time windows (■), the strength of correlations depends logarithmically on the mean trading activity of the stock, $\gamma_t(\Delta t>60\text{ min})=0.05\pm 0.01$. Shuffled data (▽) display no correlations; thus, $H_{\text{shuff}}(i)=0.5$, which also implies $\gamma_t=0$. The inset shows the log σ-log Δt scaling plot for Dell (DELL). The darker shaded intervals have well-defined Hurst exponents; the crossover is indicated with a lighter background. The slopes corresponding to Hurst exponents are 0.54 and 0.90; the slope for shuffled data is 0.50. Shuffled points were shifted vertically for better visibility.

III. STYLIZED FACTS OF TRADING ACTIVITY: SUMMARY AND NEW RESULTS

This section presents a few recent advances in understanding the empirical properties of trading activity. Their focus is on the fundamental role of company size. This is usually measured by the capitalization, but that is closely related to the trading frequency, which in turn influences a very broad range of statistical properties observed in data.

A. Size-dependent correlations

The presence of long-range autocorrelations in various measures of trading is a well-known fact [8–10]. For example, stock market volatility [1,2,12] and trading volumes [8,13] show strong persistence. Correlations can be characterized by the Hurst exponent $H(i)$ [14,15]. For stock i, this is defined[1] as

$$\sigma_i(\Delta t)=\langle[f_i^{\Delta t}(t)-\langle f_i^{\Delta t}(t)\rangle]^2\rangle \propto \Delta t^{H(i)}, \tag{3}$$

where $\langle\cdot\rangle$ denotes time averaging. There is a range of methods [15–17] to estimate the Hurst exponent, and the understanding of the results is well established [14]. The signal is said to be correlated (persistent) when $H>0.5$, uncorrelated when $H=0.5$, and anticorrelated (antipersistent) for $H<0.5$.

It is intriguing that stock market activity has a much richer behavior than simply all stocks having Hurst exponents statistically distributed around an average value, as assumed in Ref. [13]. Instead, there is a crossover [8–10] between two types of behavior around the time scale of 1 day. We located this threshold by a technique that will be discussed in Sec. IV C. An essentially uncorrelated regime was found when $\Delta t<20$ min for NYSE and $\Delta t<2$ min for NASDAQ, while the time series of larger companies become strongly correlated when $\Delta t>300$ min for NYSE and $\Delta t>60$ min for NASDAQ. As a reference, we also calculated the Hurst exponents $H_{\text{shuff}}(i)$ of the shuffled time series. The results are given in Figs. 1 and 2.[2]

One can see that for shorter time windows, correlations are absent in both markets, $H(i)\approx 0.51-0.53$. For windows longer than a trading day, however, while small $\langle f\rangle$ stocks again display only very weak correlations, larger ones show up to $H\approx 0.9$. Furthermore, there is a distinct logarithmic trend in the data:

[1]Despite earlier arguments [13], $\sigma_i(\Delta t)$ is not divergent [8,28] and so $H(i)$ can indeed be introduced.

[2]We also investigated the effect of randomly shuffling the Fourier phases of the data, which destroys the possible nonlinearities in the time series. One finds that the crossover behavior persists after such a transformation.

$$H(i) = H^* + \gamma_t \log\langle f_i \rangle, \quad (4)$$

with $\gamma_t(\Delta t > 300 \text{ min}) = 0.06 \pm 0.01$ for NYSE and $\gamma_t(\Delta t > 60 \text{ min}) = 0.05 \pm 0.01$ for NASDAQ. Shorter time scales correspond to the special case $\gamma_t = 0$; there is no systematic trend in H. Shuffled data, as expected, show $H_{\text{shuff}}(i) \approx 0.5$ at all time scales and without significant dependence on $\langle f_i \rangle$.

It is to be emphasized that the crossover is not simply between uncorrelated and correlated regimes. It is instead between homogeneous [all stocks show $H(i) \approx H_1$, $\gamma_t = 0$] and inhomogeneous ($\gamma_t > 0$) behavior. One finds $H_1 \approx 0.5$, but very small $\langle f \rangle$ stocks do not depart much from this value even for large time windows. This is a clear relation to company size, as $\langle f \rangle$ is a monotonically growing function of company capitalization [8]. Dependence of the effect on $\langle f \rangle$ is in fact a dependence on company size.

B. Fluctuation scaling of *f*

This paper will mainly focus on a special property of the time series $f_i^{\Delta t}(t)$: *fluctuation scaling* [5–7]. This connects the standard deviation σ_i and the average $\langle f_i \rangle$ of the trading activity for all our $i = 1, \ldots, N$ stocks:

$$\sigma_i(\Delta t) \propto \langle f_i \rangle^{\alpha(\Delta t)}. \quad (5)$$

Due to the long time period (3 years), the data are highly instationary. Thus, unlike a previous study [7], here we applied the DFA procedure [15,16] to estimate $\sigma_i(\Delta t)$. We determined the values of α for traded value fluctuations by fits to (5); examples are shown in Fig. 3.

The exponent α strongly depends on the size Δt of the time windows. Recently, Refs. [10,18,19] pointed out that the trading activities of NYSE and NASDAQ display very different temporal correlations, possibly due to their different trading mechanisms. Still, the scaling (5) does hold regardless of market and Δt. Furthermore, the functions $\alpha(\Delta t)$ agree qualitatively. The exponents are shown for NYSE and NASDAQ in Figs. 4(a) and 4(b), respectively. One can see that α is a nondecreasing function of Δt, and in large regimes it is, to a good approximation, either constant or logarithmic.

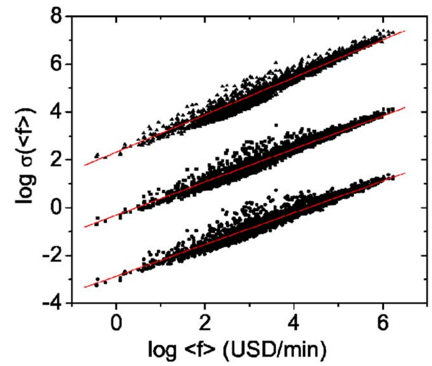

FIG. 3. (Color online) Examples of $\sigma(\langle f \rangle)$ scaling plots for NYSE, years 2000–2002. The window sizes from bottom to top: $\Delta t = 10$ sec, 0.5 day, 2 weeks. The slopes are $\alpha = 0.68, 0.71, 0.80$, respectively. Points were shifted vertically for better visibility.

C. Fluctuation scaling of *N* and *V*

One can carry out a similar analysis of other quantities; here, we limit ourself to two of those. The first one, the number of trades of stock i in size Δt time windows, will be denoted by $N_i^{\Delta t}(t)$, its variance by $\sigma_N^2(i, \Delta t)$. The second one was introduced before; $V_i(n)$ is the value exchanged in the nth trade of stock i. The corresponding variance will be σ_{Vi}^2.

Dimensional analysis predicts

$$\sigma_{Vi}^2 \propto \langle V_i \rangle^2, \quad (6)$$

which is remarkably close to the observed behavior, shown

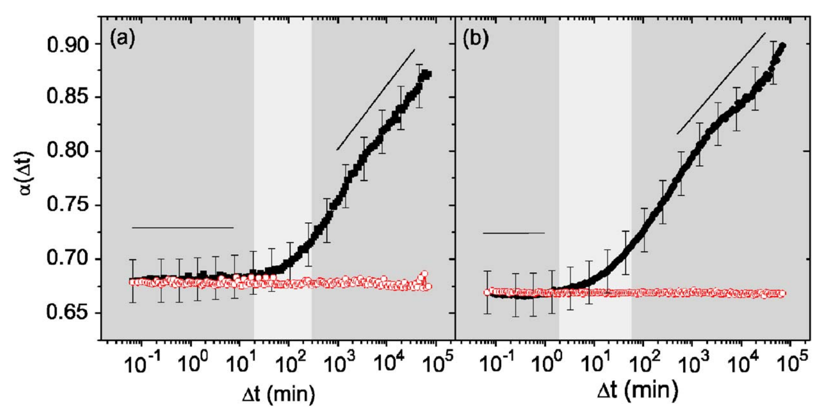

FIG. 4. (Color online) The dependence of the scaling exponent α on the window size Δt. The darker shaded intervals have well-defined Hurst exponents and values of γ_t; the crossover is indicated with a lighter background. (a) NYSE: without shuffling (■) the slopes of the linear regimes are $\gamma_f(\Delta t < 20 \text{ min}) = 0.00 \pm 0.01$ and $\gamma_f(\Delta t > 300 \text{ min}) = 0.06 \pm 0.01$. For shuffled data (○) the exponent is independent of window size, $\alpha(\Delta t) = 0.68 \pm 0.02$. (b) NASDAQ: without shuffling (■) the slopes of the linear regimes are $\gamma_f(\Delta t < 2 \text{ min}) = 0.00 \pm 0.01$ and $\gamma_f(\Delta t > 60 \text{ min}) = 0.06 \pm 0.01$. For shuffled data (○) the exponent is independent of window size, $\alpha(\Delta t) = 0.67 \pm 0.02$. Note that there is a deviation from linearity around $\Delta t \approx 1$ trading week. It is larger for NASDAQ, but it is still between the error bars. A possible cause is the weekly periodic pattern of trading which was not removed manually.

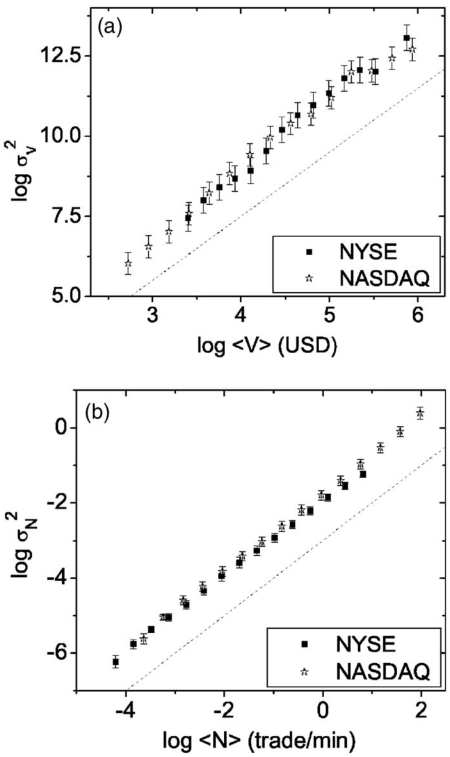

FIG. 5. (a) Plot verifying the validity of (6) for stock market data; typical error bars are given. The straight line would correspond to $\sigma_{Vi}^2 \propto \langle V_i \rangle^2$. (b) Plot verifying the validity of (7) for stock market data; typical error bars are given. The straight line would correspond to $\sigma_{Ni}^2 \propto \langle N_i \rangle$. The size of the time windows is $\Delta t = 1$ sec.

in Fig. 5(a). Also, when the size of the time windows is chosen sufficiently small ($\Delta t \ll 1$ min), the probability that two trades of the same stock happen in the same period is negligible. In this limit, correlations between consecutive trades cannot contribute to σ_N^2, the central limit theorem becomes applicable, and simply

$$\sigma_{Ni}^2 \propto \langle N_i \rangle, \tag{7}$$

which again agrees very well with empirical data shown for $\Delta t = 1$ sec in Fig. 5(b).

D. Dependence of typical trade size on trading frequency

The final observation to be discussed here is that for a large group of stocks; the average rate of trades $\langle N \rangle$ and their mean value $\langle V \rangle$ are connected by a power law:

$$\langle V_i \rangle \propto \langle N_i \rangle^{\beta}. \tag{8}$$

Such relationships are shown in Figs. 6(a) and 6(b) for NYSE and NASDAQ, respectively. The measured exponents are $\beta_{\mathrm{NYSE}} = 0.59 \pm 0.09$ and $\beta_{\mathrm{NASDAQ}} = 0.22 \pm 0.04$, although they are restricted to large enough stocks. The estimate based on Ref. [20] for the stocks in London's FTSE-100, is $\beta \approx 1$.

The values of β_{NYSE} and β_{NASDAQ}, and especially the marked difference between them, appear to be very robust for various time periods. One major contribution to this is

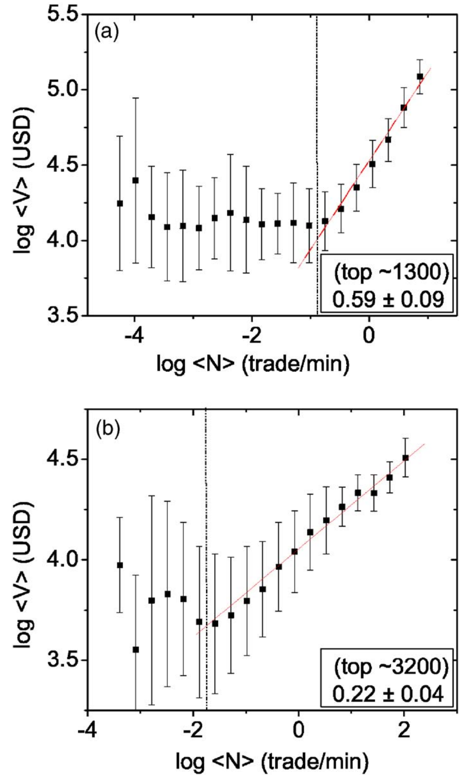

FIG. 6. (Color online) The dependence of the mean value per trade $\langle V_i \rangle$ on the average rate of trades $\langle N_i \rangle$. Calculations were done for the period 2000–2002. (a) shows NYSE and (b) shows NASDAQ. Points were binned and their logarithm was averaged for better visibility; error bars show the standard deviations in the bins. For the smallest stocks there is no clear trend at either exchange. However, larger stocks at NYSE and all except the minor ones at NASDAQ show scaling between the two quantities, equivalent to that given in (8). The slopes are $\beta_{\mathrm{NYSE}} = 0.59 \pm 0.04$ and $\beta_{\mathrm{NASDAQ}} = 0.22 \pm 0.04$.

probably the difference in trading mechanisms between the two markets [10,21].

One very crude interpretation of the effect in general is the following. Smaller stocks are exchanged rarely, but transaction costs must limit from below the value that is still profitable to be exchanged at once. This minimal unit is around the order of 10^4 US dollars for both markets. Once the speed of trading and liquidity grow, it becomes possible to exchange larger packages. Trades start to "stick together"; their average value starts to grow. Although this tendency reduces transaction costs, the price impact [22–25] of the trade also increases, which in practice often limits package sizes from above. These two mechanisms may have a role in the formation of (8). Also, as they vary strongly from market to market, such very different values of β might be justified.

IV. SCALING THEORY

In this section, we present a framework that unifies the—seemingly unrelated—observations of Sec. III. This is centered around the above-introduced fluctuation scaling (5):

$$\sigma_i(\Delta t) \propto \langle f_i \rangle^{\alpha(\Delta t)}.$$

This phenomenon is not at all specific to stock market data; in fact, it has been observed for activity in a wide range of complex systems. Possible choices for f include data traffic through Internet routers, daily webpage hits, highway traffic [5,6], and node visitations of random walkers on complex networks [5,26]. In this sense, the stock market is seen as a complex system, where the constituents are stocks and their activity at any time is given by the traded value per unit time.

A. Universal values of α

First, notice that (3) and (5) are formal analogs. They connect the same standard deviation with the two complementary factors: the Δt size of the time window and the average (trading) activity $\langle f_i \rangle$. There is evidence that while $H(i)$ describes the correlation properties of the individual elements activity, the function $\alpha(\Delta t)$ carries information about the collective dynamical properties of the whole system. Based on this knowledge, a classification scheme was outlined in Refs. [5,26,27]. All those studies assume that the activities of all nodes are uncorrelated—i.e., $H(i)=0.5$.[3] In this case, there are two known universality classes with respect to the value of α.

In certain systems, the activity of the constituents comes from nearly equivalent, independent events. The difference between nodes with smaller and greater mean activity comes basically from the different mean *number* of events. Then, the *central limit theorem* can be applied to these events and this yields $\alpha=0.5$ automatically. Examples include simple surface growth models and the data traffic of Internet routers [5].

Other systems dynamics are under a *dominant external driving force*: Activity fluctuations are mainly caused by the variations of this external force, and this leads to proportionality between the strength and standard deviations at the nodes: $\alpha=1$, regardless of the internal structure or the laws governing the time evolution. This is observed for the statistics of webpage visitations and highway traffic [5].

In temporally uncorrelated systems, two processes are known to give rise to intermediate measured values $0.5<\alpha<1$: Some finite systems display a crossover between $\alpha=0.5$ and $\alpha=1$ at a certain node strength $\langle f \rangle$, due to the competition of external driving and internal dynamics [5,6]. There is an *effective* value of α, but in fact, scaling breaks down. Another possible scenario is discussed in the following.

B. Nonuniversal values of α

The activities $f_i(t)$ originate from individual events that take place at the nodes. Every event n at node i is characterized by its time $t_i(n)$ and its size $V_i(n)$ which is now allowed to vary. For a given size of time windows Δt, the observed time series is given by

$$f_i^{\Delta t}(t) = \sum_{n, t_i(n) \in [t, t+\Delta t]} V_i(n),$$

a formula equivalent to (1). In the stock market, the value exchanged in a trade is a plausible choice of V.

If the random process that gives the size of an event is independent of the one that determines when the event occurs, one can find a simple formula [26] that shows how fluctuations of f are composed:

$$\sigma_i^2 = \sigma_{Vi}^2 \langle N_i \rangle + \sigma_{Ni}^2 \langle V_i \rangle^2, \tag{9}$$

where $\langle V_i \rangle$ and σ_{Vi}^2 are the mean and the standard deviations of the event size distribution. $\langle N_i \rangle$ and σ_{Ni}^2 are similar, only for the number of events in time windows of length Δt. Under these conditions, it is also trivial that $\langle f_i \rangle = \langle N_i \rangle \langle V_i \rangle$.

All the above can be expected from simple principles. Two more relationships are necessary and are often realized; they are basically the same as (6) and (7). The only strong assumption to account for nonuniversal values of α is the following. Consider a system where elements with higher average activity do not only experience more events, but those are also larger. Let us assume scaling between the two quantities:

$$\langle V_i \rangle \propto \langle N_i \rangle^{\beta},$$

which is equivalent to (8). Then, α can be expressed [26], by combining all the formulas, as

$$\alpha = \frac{1}{2}\left(1 + \frac{\beta}{\beta+1}\right). \tag{10}$$

In this general context, the property $\beta \neq 0$ can be called *event size inhomogeneity*.[4] The intermediate values $0.5<\alpha<1$ interpolate between the square root type of $\langle N \rangle \propto \sigma_N^{1/2}$ and the linear $\langle V \rangle \propto \sigma_V$, while the conditions ensure that scaling is preserved [compare Figs. 3, 5(a), and 5(b)].

These conditions are satisfied exactly in a random walker model on complex networks [26]. Consequently, its behavior is well described by (10). However, such arguments can also be applied to stock market trading dynamics when $\Delta t \ll 1$ min to ensure the validity of (7). By substituting the observed values of β, one finds the estimates $\alpha^*_{\text{NYSE}}=0.69\pm0.03$ and $\alpha^*_{\text{NASDAQ}}=0.59\pm0.02$. The actual values are $\alpha_{\text{NYSE}}(\Delta t\to 0)=0.68\pm0.02$ and $\alpha_{\text{NASDAQ}}(\Delta t\to 0)=0.67\pm0.02$. The agreement for the NYSE data is good; for NASDAQ, it is only approximate. Moreover, Eq. (8) only fits the data for large enough stocks, while Eq. (5) gives an excellent fit over the whole range available. Therefore, this explanation is only partial; however, it indicates that $\alpha>0.5$ is to be expected. This is a crucial point, because markets are so far the only examples of an $0.5<\alpha<1$ system.

C. Time scale dependence of α

Section III revealed that the exponent α of stock market activity fluctuations shows a strong dependence on the time

[3]Note that, in general, instead of uncorrelated dynamics, it is enough if the activity of every node displays the same Hurst exponent $H(i)=H$. This is the direct consequence of arguments in Sec. IV.

[4]In Refs. [26,27], $V_i(n)$ is called the *impact* and such a property is called the *impact inhomogeneity*. However, here we wish to avoid confusion with another financial term, price impact [22–25].

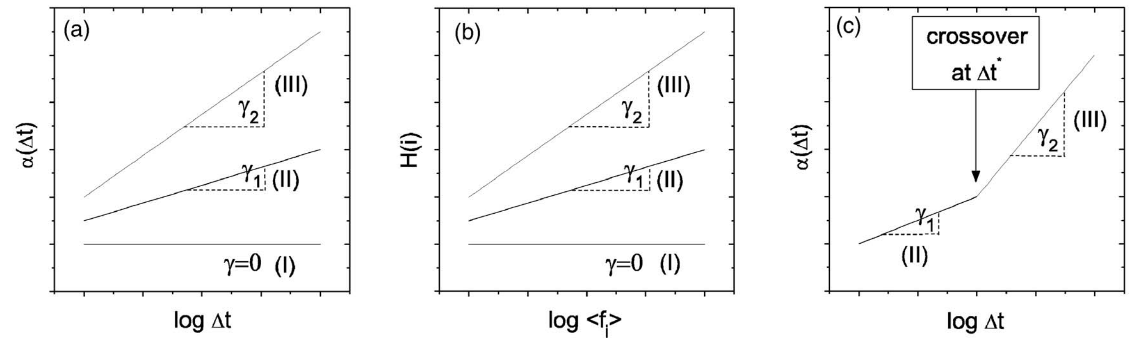

FIG. 7. (a), (b) Possible scenarios where both $\sigma_i(\Delta t) \propto \Delta t^{H(i)}$ and $\sigma_i(\Delta t) \propto \langle f_i \rangle^{\alpha(\Delta t)}$ can be satisfied simultaneously. (I) In systems where $\gamma=0$, α is independent of window size and H is independent of node. (II) When $\gamma=\gamma_1>0$, $\alpha(\Delta t)$ and $H(i)$ depend logarithmically on Δt and on $\langle f_i \rangle$, respectively, with the common slope γ_1. (III) For a larger value, $\gamma=\gamma_2>\gamma_1$, the dependence is stronger. (c) Example of a crossover between different values of γ. There, α still depends on Δt in a logarithmic way, but the slope is different in two regimes. In this case, for every node there are two Hurst exponents $H_1(i)$ and $H_2(i)$ that are valid asymptotically for $\Delta t \ll \Delta t^*$ and $\Delta t \gg \Delta t^*$, respectively. Then, both of these must independently follow the logarithmic law shown in (b): $H_1(i)=H_1^*+\gamma_1 \log\langle f_i \rangle$ and $H_2(i)=H_2^*+\gamma_2 \log\langle f_i \rangle$.

window Δt. This was previously attributed to the effect of external factors [7]. On the time scale of minutes, news, policy changes, etc., have no time to diffuse in the system. Thus, temporal fluctuations are dominated by internal dynamics, $\alpha<1$. By increasing Δt to days or weeks, the importance of this external influence grows and α approaches 1, which is characteristic in the presence of a strong external driving. However, the effect just described is a crossover, while observations show the persistence of scaling; only the exponent α changes. This section offers an alternative description that has no such shortcoming.

The key is to extend the analysis to $H(i)\neq 0.5$ systems. We start from the relations (3) and (5), where the role of the two variables $\langle f_i \rangle$ and Δt is analogous. When they hold simultaneously, from the equality of their left-hand sides, one can write the third proportionality

$$\Delta t^{H(i)} \propto \langle f_i \rangle^{\alpha(\Delta t)}.$$

After taking the logarithm of both sides, differentiation $\partial^2/\partial(\log \Delta t)\partial(\log\langle f_i \rangle)$ yields the asymptotic equality

$$\gamma_t \sim \frac{dH(i)}{d(\log\langle f_i \rangle)} \sim \frac{d\alpha(\Delta t)}{d(\log \Delta t)} \sim \gamma_f. \qquad (11)$$

This means that both partial derivatives are constant and they have the same value, which we will denote by $\gamma=\gamma_t=\gamma_f$.

The possibilities how this can be realized are sketched in Figs. 7(a) and 7(b).

(I) In systems where $\gamma=0$, the exponent $\alpha(\Delta t)=\alpha^*$, it is independent of window size. At the same time all nodes must exhibit the same degree of correlations, $H(i)=H$.

(II) In the case when $\gamma=\gamma_1>0$, $\alpha(\Delta t)$ actually depends on Δt. This dependence must be logarithmic: $\alpha(\Delta t)=\alpha^* +\gamma_1 \log \Delta t$. At the same time, the Hurst exponent of the nodes depends on the mean flux in a similar way: $H(i)=H^* +\gamma_1 \log\langle f_i \rangle$. Moreover, the slope of the logarithmic dependence is the same.

(III) When the constant γ is larger—for example, $\gamma_2 >\gamma_1$ in Figs. 7(a) and 7(b)—α changes faster with Δt, while also $H(i)$ changes faster with $\langle f_i \rangle$.

Finally, a combination of these options is also possible. Systems may display a crossover between different values of γ at a certain time scale Δt^*; an example is given in Figs. 7(b) and 7(c). There, α depends on Δt in a logarithmic way, but the slope of the trend is different in two regimes. In this case, there is no unique Hurst exponent of $f_i(t)$. Instead, for every node there are two values $H_1(i)$ and $H_2(i)$, which are valid asymptotically, for $\Delta t \ll \Delta t^*$ and $\Delta t \gg \Delta t^*$, respectively. Then, both of these must independently follow the logarithmic law $H_1(i)=H_1^*+\gamma_1 \log\langle f_i \rangle$ and $H_2(i)=H_2^*+\gamma_2 \log\langle f_i \rangle$.

Stock markets belong to this last group. For $\Delta t \leq 20$ min for NYSE and $\Delta t \leq 2$ min for NASDAQ, $\alpha(\Delta t)\approx \alpha^*$. Correspondingly, H must be independent of $\langle f \rangle$, as was found in Sec. III. On the other hand, for $\Delta t>300$ min for NYSE and $\Delta t>60$ min for NASDAQ, $\alpha(\Delta t)$ is approximately logarithmic with the common coefficient $\gamma=0.06\pm0.01$. This, again, must equal the slope of $H(i)$ plotted versus $\log\langle f_i \rangle$. There is agreement between error bars with the results of Sec. III.

The fact that the local derivative $\frac{d\alpha(\Delta t)}{d(\log \Delta t)}$ also shows the degree of logarithmic trend in the Hurst exponents gives a visual method to detect the change in this collective behavior of the market. Those regimes in Δt, where $\alpha(\Delta t)$ is constant, correspond to time scales where all stocks have the same level (Hurst exponent) of activity correlations. Where $\alpha(\Delta t)$ is logarithmically changing, the slope γ gives the degree of inhomogeneity in $H(i)$. Finally, the function is curved near crossovers, where the degree of the mean flux dependence in correlation strengths is changing.

In order to underline that the $\alpha(\Delta t)$ dependence comes from temporal correlations, we carried out the same measurement, but with all time series shuffled randomly. It is trivial that if Δt equals the $\delta=1$ sec resolution of the data set, shuffling does not affect the estimates of $\sigma_i(\Delta t=\delta)$; it merely rearranges the terms used in averaging.[5] Hence, the fitted slope cannot change either, $\alpha_{\text{shuff}}(\delta)=\alpha(\delta)$. On the other

[5]In fact, the DFA procedure can only be applied for $\Delta t \geq 4\delta$, but the effect of this difference is negligible.

hand, shuffling gives uncorrelated time series $H_{\text{shuff}}(i) \equiv 0.5$ (see Sec. III). Correspondingly, $\gamma_{\text{shuff}} = \frac{dH_{\text{shuff}}}{d \log\langle f \rangle} = 0$. Hence, according to (11), $\alpha_{\text{shuff}}(\Delta t) = \alpha^*$, regardless of window size. The measurement results—in excellent agreement with the above reasoning—are shown by open circles in Figs. 4(a) and 4(b).

Finally, we must emphasize that the value of γ is not *a priori* known for real systems. Consequently, α does not reflect the type of internal dynamics in the straightforward fashion suggested by Ref. [5]. Instead, a careful analysis, including the dependence on Δt, must be undertaken in order to interpret the results correctly. The only exception is when we can assume homogeneous correlations—i.e., $\gamma = 0$ and so $H(i) = H$.

V. CONCLUSIONS

In the above, we generalized a fluctuation scaling relation to the case when temporal correlations are present in the individual time series. In such an analysis, one measures the time-scale-dependent scaling exponent $\alpha(\Delta t)$. In addition to previous studies, we found that even in the presence of strong temporal correlations, α still remains very characteristic to the internal dynamics. Indeed, its time scale dependence reveals additional information. For the persistence of fluctuation scaling at all time scales, it is inevitable that the strength of correlations in *all* the individual time series be connected by a logarithmic law. Such a relationship is a peculiar feature of collective dynamics, which is not explained by the number or size distribution of the events that occur at the nodes.

The framework was applied to reveal the connections between stylized facts of stock market trading activity. Empirical data for both of the markets NYSE and NASDAQ show qualitatively similar behavior. The values of $\alpha(\Delta t)$ can be understood based on the role of company size. For short times when there are no correlations between the trades of an individual company, the nontrivial value of α comes from the highly inhomogeneous trade sizes of the different companies. For increasing time windows, we observe a logarithmic law in correlation strengths and that this leads to a window size dependence of α. As the growing size of individual trades with increasing company size can also be considered as a cumulation of smaller transactions, our results underline the importance of temporal correlations and size dependence in explaining scaling phenomena on the stock market.

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[1] J.-P. Bouchaud and M. Potters, *Theory of Financial Risk* (Cambridge University Press, Cambridge, England, 2000).
[2] R. N. Mantegna and H. E. Stanley, *Introduction to Econophysics: Correlations and Complexity in Finance* (Cambridge University Press, Cambridge, England, 1999).
[3] *Disordered and Complex Systems*, AIP Conf. Proc. No. 553, edited by Peter Sollich, A. C. C. Coolen, L. P. Hughston, and R. F. Streater (AIP, Melville, NY, 2001).
[4] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[5] M. A. de Menezes and A.-L. Barabási, Phys. Rev. Lett. **92**, 028701 (2004).
[6] M. A. de Menezes and A.-L. Barabási, Phys. Rev. Lett. **93**, 068701 (2004).
[7] Z. Eisler, J. Kertész, S.-H. Yook, and A.-L. Barabási, Europhys. Lett. **69**, 664 (2005).
[8] Z. Eisler and J. Kertész, e-print physics/0508156, Eur. Phys. J. B (to be published).
[9] P. Ch. Ivanov, A. Yuen, B. Podobnik, and Y. Lee, Phys. Rev. E **69**, 056107 (2004).
[10] A. Yuen and P. Ch. Ivanov, e-print physics/0508203.
[11] Trades and Quotes Database for 2000–2002, New York Stock Exchange, New York.
[12] R. Cont, Quant. Finance **1**, 223 (2001).
[13] P. Gopikrishnan, V. Plerou, X. Gabaix, and H. E. Stanley, Phys. Rev. E **62**, R4493 (2000).
[14] T. Vicsek, *Fractal Growth Phenomena* (World Scientific, Singapore, 1992).
[15] J. W. Kantelhardt, S. A. Zschiegner, E. Koscielny-Bunde, S. Havlin, A. Bunde, and H. E. Stanley, Physica A **316**, 87 (2002).
[16] C.-K. Peng, S. V. Buldyrev, S. Havlin, M. Simons, H. E. Stanley, and A. L. Goldberger, Phys. Rev. E **49**, 1685 (1994).
[17] J. F. Muzy, E. Bacry, and A. Arneodo, Phys. Rev. Lett. **67**, 3515 (1991).
[18] A. G. Zawadowski, G. Andor, and J. Kertész, Physica A **344**, 221 (2004).
[19] A. G. Zawadowski, G. Andor, and J. Kertész, e-print cond-mat/0406696, Quant. Finance (to be published).
[20] G. Zumbach, Quant. Finance **4**, 441 (2004).
[21] Data available upon request.
[22] X. Gabaix, P. Gopikrishnan, V. Plerou, and H. E. Stanley, Nature (London) **423**, 267 (2003).
[23] V. Plerou, P. Gopikrishnan, X. Gabaix, and H. E. Stanley, Quant. Finance **4**, C11 (2004).
[24] J. D. Farmer and F. Lillo, Quant. Finance **4**, C7 (2004).
[25] J. D. Farmer, L. Gillemot, F. Lillo, S. Mike, and A. Sen, Quant. Finance **4**, 383 (2004).
[26] Z. Eisler and J. Kertész, Phys. Rev. E **71**, 057104 (2005).
[27] J. Kertész and Z. Eisler, in Practical Fruits of Econophysics: Proceedings of the Third Nikkei Econophysics Symposium, edited by Hideki Takayasu (Springer, Tokyo, 2005), e-print physics/0503139.
[28] S. M. D. Queirós, Europhys. Lett. **71**, 339 (2005).